Solutions of a Fractional Algebraic Equation

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Abstract: **In this paper, based on a new multiplication of fractional analytic functions, we solve a fractional algebraic equation. The solutions of this fractional algebraic equation can be obtained by using some methods. In fact, the solutions are generalizations of the traditional algebraic equation solutions.**

Keywords: **New multiplication, Fractional analytic functions, Fractional algebraic equation.**

I. INTRODUCTION

Fractional calculus is the theory of derivative and integral of non-integer order, which can be traced back to Leibniz, Liouville, Grunwald, Letnikov and Riemann. Fractional calculus has been attracting the attention of scientists and engineers from long time ago, and has been widely used in physics, mechanics, control theory, viscoelasticity, electrical engineering, biology, economics and other fields [1-13].

In this paper, based on a new multiplication of fractional analytic functions, we study a fractional algebraic equation. Using some techniques, we can find the solutions of this fractional algebraic equation. Moreover, our result is a generalization of the result of ordinary algebraic equation.

II. PRELIMINARIES

Definition 2.1 ([14]): Suppose that x and a_k are real numbers for all k, and $0 < \alpha \leq 1$. If the function $f_\alpha: [a, b] \rightarrow R$ can be expressed as an α-fractional power series, that is, $f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)} x^{k\alpha}$, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic function.

In the following, we introduce a new multiplication of fractional analytic functions.

Definition 2.2 ([15]): If $0 < \alpha \leq 1$. Suppose that $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional analytic functions,

$$f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)} x^{k\alpha} = \sum_{k=0}^{\infty} \frac{a_k}{k!} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha \right)^{\otimes k}, \tag{1}$$

$$g_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)} x^{k\alpha} = \sum_{k=0}^{\infty} \frac{b_k}{k!} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha \right)^{\otimes k}. \tag{2}$$

Then

$$f_\alpha(x^\alpha) \otimes g_\alpha(x^\alpha)$$

$$= \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)} x^{k\alpha} \otimes \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)} x^{k\alpha}$$

$$= \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)} \left(\sum_{m=0}^{k} \binom{k}{m} a_{k-m} b_m \right) x^{k\alpha}. \tag{3}$$

Equivalently,

$$f_\alpha(x^\alpha) \otimes g_\alpha(x^\alpha)$$

$$= \sum_{k=0}^\infty \frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes k} \otimes \sum_{k=0}^\infty \frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes k}$$

$$= \sum_{k=0}^\infty \frac{1}{k!}\left(\sum_{m=0}^k \binom{k}{m} a_{k-m}b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes k}. \tag{4}$$

Definition 2.3 ([16])**:** If $0 < \alpha \le 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions. Then $\left(f_\alpha(x^\alpha)\right)^{\otimes n} = f_\alpha(x^\alpha) \otimes \cdots \otimes f_\alpha(x^\alpha)$ is called the nth power of $f_\alpha(x^\alpha)$. Moreover, if $f_\alpha(x^\alpha) \otimes g_\alpha(x^\alpha) = 1$, then $g_\alpha(x^\alpha)$ is called the $\otimes$ reciprocal of $f_\alpha(x^\alpha)$, and is denoted by $\left(f_\alpha(x^\alpha)\right)^{\otimes -1}$.

III.   MAIN RESULT

In this section, we solve a fractional algebraic equation.

Problem 3.1: Let $0 < \alpha \le 1$, and $(-1)^{\frac{1}{\alpha}}$ exists. Find the solutions of the 4-th order α–fractional algebraic equation

$$2 \cdot \left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes 4} + \left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes 2} \otimes \left[3 \cdot \frac{1}{\Gamma(\alpha+1)}x^\alpha + 1\right] - 4 \cdot \left[3 \cdot \frac{1}{\Gamma(\alpha+1)}x^\alpha + 1\right]^{\otimes 2} = 0. \tag{5}$$

Solution Both sides of this equation are divided by $\left[3 \cdot \frac{1}{\Gamma(\alpha+1)}x^\alpha + 1\right]^{\otimes 2}$, then

$$2 \cdot \left[\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes 2} \otimes \left[3 \cdot \frac{1}{\Gamma(\alpha+1)}x^\alpha + 1\right]^{\otimes -1}\right]^{\otimes 2} + \left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes 2} \otimes \left[3 \cdot \frac{1}{\Gamma(\alpha+1)}x^\alpha + 1\right]^{\otimes -1} - 4 = 0. \tag{6}$$

Let $\frac{1}{\Gamma(\alpha+1)}t^\alpha = \left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes 2} \otimes \left[3 \cdot \frac{1}{\Gamma(\alpha+1)}x^\alpha + 1\right]^{\otimes -1}$, we have

$$2 \cdot \left[\frac{1}{\Gamma(\alpha+1)}t^\alpha\right]^{\otimes 2} + \frac{1}{\Gamma(\alpha+1)}t^\alpha - 4 = 0. \tag{7}$$

That is,

$$\frac{4}{\Gamma(2\alpha+1)}t^{2\alpha} + \frac{1}{\Gamma(\alpha+1)}t^\alpha - 4 = 0. \tag{8}$$

Therefore,

$$t^\alpha = \frac{-\frac{1}{\Gamma(\alpha+1)} \pm \sqrt{\left[\frac{1}{\Gamma(\alpha+1)}\right]^2 + \frac{64}{\Gamma(2\alpha+1)}}}{\frac{8}{\Gamma(2\alpha+1)}}$$

$$= \frac{-\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} \pm \sqrt{\left[\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}\right]^2 + 64 \cdot \Gamma(2\alpha+1)}}{8}$$

$$= \frac{-\Gamma(2\alpha+1) \pm \sqrt{[\Gamma(2\alpha+1)]^2 + 64 \cdot [\Gamma(\alpha+1)]^2 \cdot \Gamma(2\alpha+1)}}{8 \cdot \Gamma(\alpha+1)}. \tag{9}$$

Hence,

$$\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes 2} \otimes \left[3 \cdot \frac{1}{\Gamma(\alpha+1)}x^\alpha + 1\right]^{\otimes -1}$$

$$= \frac{1}{\Gamma(\alpha+1)}t^\alpha$$

$$= \frac{-\Gamma(2\alpha+1) \pm \sqrt{[\Gamma(2\alpha+1)]^2 + 64 \cdot [\Gamma(\alpha+1)]^2 \cdot \Gamma(2\alpha+1)}}{8 \cdot [\Gamma(\alpha+1)]^2}. \tag{10}$$

Let $\lambda_{1,2} = \frac{-\Gamma(2\alpha+1) \pm \sqrt{[\Gamma(2\alpha+1)]^2 + 64 \cdot [\Gamma(\alpha+1)]^2 \cdot \Gamma(2\alpha+1)}}{8 \cdot [\Gamma(\alpha+1)]^2}$, then

$$\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes 2} \otimes \left[3 \cdot \frac{1}{\Gamma(\alpha+1)}x^\alpha + 1\right]^{\otimes -1} = \lambda_{1,2} \ . \tag{11}$$

We obtain

$$\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes 2} - 3 \cdot \lambda_{1,2} \cdot \frac{1}{\Gamma(\alpha+1)}x^\alpha - \lambda_{1,2} = 0 \ . \tag{12}$$

That is,

$$\frac{2}{\Gamma(2\alpha+1)}x^{2\alpha} - 3 \cdot \lambda_{1,2} \cdot \frac{1}{\Gamma(\alpha+1)}x^\alpha - \lambda_{1,2} = 0 \ . \tag{13}$$

Thus,

$$x^\alpha = \frac{3 \cdot \lambda_{1,2} \pm \sqrt{9 \cdot \lambda_{1,2}{}^2 + \frac{8}{\Gamma(2\alpha+1)} \cdot \lambda_{1,2}}}{\frac{4}{\Gamma(2\alpha+1)}}$$

$$= \frac{3 \cdot \Gamma(2\alpha+1) \cdot \lambda_{1,2} \pm \sqrt{9 \cdot [\Gamma(2\alpha+1)]^2 \cdot \lambda_{1,2}{}^2 + 8 \cdot \Gamma(2\alpha+1) \cdot \lambda_{1,2}}}{4} \ . \tag{14}$$

Finally, we get

$$x = \left[\frac{3 \cdot \Gamma(2\alpha+1) \cdot \lambda_{1,2} \pm \sqrt{9 \cdot [\Gamma(2\alpha+1)]^2 \cdot \lambda_{1,2}{}^2 + 8 \cdot \Gamma(2\alpha+1) \cdot \lambda_{1,2}}}{4}\right]^{\frac{1}{\alpha}} \ . \tag{15}$$

IV. CONCLUSION

In this paper, based on a new multiplication of fractional analytic functions, we solve a fractional algebraic equation. In fact, the fractional algebraic equation is a generalization of classical algebraic equation. By some techniques, we can find the solutions of this fractional algebraic equation. In the future, we will continue to use the new multiplication of fractional analytic functions to solve the problems in engineering mathematics and fractional calculus.

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